"∞-categories" $\Longleftrightarrow$ "quasicategories"

Motivation:

- Higher groupoids. (Grothendieck)

  · Sets $X(R) = \{(x,y,z) \in R^3 \mid x^3+y^3=z^3\}$

  · **Ex**: X = top'l space, G = group

  $\Rightarrow \{\text{principal } G\text{-bundles on } X\}/_{\text{iso}} = ??$

  $\hookrightarrow \mathrm{Bun}_G(X)$ = groupoid (= 1-groupoid)

  $= \begin{cases} \textbf{objects}: & P \to X \text{ principal } G\text{-bundle} \\ \textbf{mor}: & \begin{tikzcd} P \arrow[d, "="'] \arrow[dr] & \\ P' \arrow[r] & X \end{tikzcd} \text{ equiv.} \end{cases}$

$X = U \cup V$

$$\begin{tikzcd} \mathrm{Bun}_G(X) \arrow[r] \arrow[d] & \mathrm{Bun}_G(U) \arrow[d] \\ \mathrm{Bun}_G(V) \arrow[r] & \mathrm{Bun}_G(U \cap V) \end{tikzcd}$$

"homotopy" pullback of groupoids

$$\mathrm{Bun}_G(X)/_{\simeq} \simeq H^1(X; G)$$

$\Rightarrow$ n-groupoids $\Leftarrow$ $H^n(X,A)$

Ex: $\{$1-groupoids$\}$ $\in$ 2-groupoid

↳ objects = groupoids

morphism = functors

2-morphism = nat isos of functors

Grothendieck: $\{\infty$-groupoids$\}$ $\Leftrightarrow$ $\{$homotopy types of spaces$\}$

"Homotopy Hypothesis" $\quad \rotatebox{90}{$\subseteq$}$

$\{$1-groupoids$\}$ $\Leftrightarrow$ $\{$1-truncated spaces$\}$

classifying space $\quad G \longmapsto BG$, $\quad \xleftarrow{\pi_1}$

Combinatorial homotopy theory = simplicial sets

$\rotatebox{90}{$\subseteq$}$

"$\{$homotopy types$\}$" $\Longleftrightarrow$ $\{$Kan complexes$\}$

$\Updownarrow$ $\qquad\qquad f$

$\{\infty$-groupoids$\}$

(

· abstract homotopy theory :

- spaces + homotopies ⟹ hTop ← "homotopy category"
- chain complexes of Mod_R + chain homotopy ⟹ $D(R)$ ← "derived category"
- Kan complex + simplicial htpy ⟹ hKan ≃ hTop ⟸

Quillen: Model category ⟹ homotopy category

↳ homotopy limits + colimits

Ex:

$$\begin{tikzcd} U\cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X = U\cup V \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$\begin{tikzcd} F = E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \{*\} = B' \arrow[r] & B \end{tikzcd}$$

p.b

fiber bundle

Mayer-Vietoris seq in H_* ⟹ ho. pushout of spaces

l.e.s in π_* ⟸ ho pullback of spaces

↳ [homotopy limits/colimits are "derived functors" of ord limits/colimits

→ Model categ ⟹ ∞-category

Span, ch cpx,

Spectra, ..., ; $\in$ ∞-categories

Cat_∞

What is an ∞-category? = <u>quasicategory</u>

Boardman-Vogt 1972 ("weak Kan complexes")

A. Joyal ~ 2003 (∞-groupoids (= quasicats s.t. all maps are isos.))

$\shortparallel$

Kan complex)

J. Lurie ~2006

~ 900 pages (Higher Topos Theory)

– 1200 pages (Higher Algebra)

Cat_1

$\cap$

$\shortparallel$

2-category: $\subseteq$ bicategory

obj

1-mor

2-mor

$\cup$

2 groupoid

(all 1-mor, 2mor invertible)

∞ groupoid $\supseteq$

∞-cat = (∞,1)-category.

quasicategory

- · objects
- · 1-mor
- · 2-mor
- · 3-mor
- · 4-mor

} invertible

1-categories

(∞,2)-cats.

- · object
- · 1-mor
- 2-mor
- 3-mor
- 4-mor

} invertible

2-categories

objects

∞-cats

mor_funs

— $Cat_{(\infty,1)} = Cat_\infty$ ∈ (∞,1)-category

$\overline{Cat}_{(\infty,1)}$ ∈ (∞,2)-category

- simplicial sets — nerve
- define quasicats —